

# Weak Monotone Comparative Statics

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Virtual Market Design Seminar

# Motivation

- **Comparative Statics:** how predicted behavior changes as environment changes.
- **Monotone Comparative Statics:** [Topkis \(1979, 1998\)](#) and [Milgrom and Shannon \(1994\)](#) provide a method that captures essential properties driving comparative statics.
  - ▶ Since predictions are often nonunique, set order matters.
  - ▶ Existing theory uses strong set order

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- **Weak Set Order:**  $X'' \geq_{ws} X'$  if
  - ▶  $X'' \geq_{uws} X' : \forall x' \in X',$  there exists  $x'' \in X''$  with  $x'' \geq x'$ .
  - ▶  $X'' \geq_{lws} X' : \forall x'' \in X'',$  there exists  $x' \in X'$  with  $x' \leq x''$ .
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- Strong set order implies weak set order.

- Let  $M(f) := \arg \max_{x \in X} f(x).$  Given payoff function  $u, v,$   
 $M(v) \geq_{ss} M(u)$  if  $v$  **MS-dominates**  $u:$ 
  - ▶  $v$  *single-crossing dominates*  $u;$
  - ▶  $u, v$  are *quasi-supermodular*
- But beyond individual choices, MCS is difficult to achieve in the strong set order (e.g., social choice, games, and matching)

## Illustration with Nash equilibria

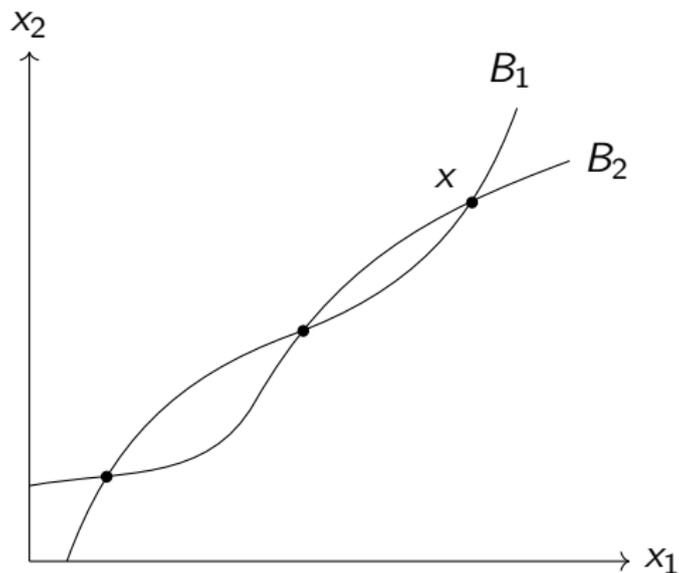


Figure: Failure of sMCS.

The MS conditions for payoffs guarantee monotonicity of best response.

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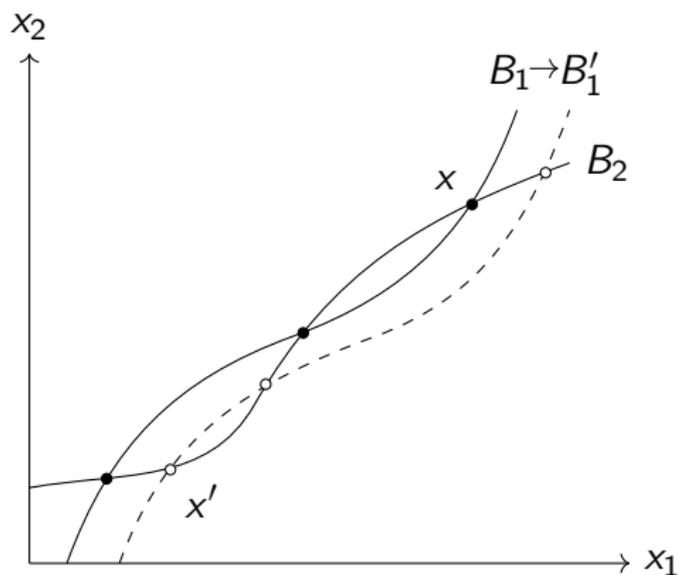


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But equilibria do not shift in the strong set order. They do shift monotonically in the weak set order.

# What We Do

- We consider **weak monotone comparative statics (wMCS)**
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- Look for conditions for wMCS in the context of:
  - ▶ Individual choices
  - ▶ Pareto optimal choices
  - ▶ Games
  - ▶ Two-sided matching
- In the process, we make progress on
  - ▶ existence of fixed points and Nash equilibria in games
  - ▶ characterization and existence of stable matching in two-sided matching
- Expand applications of game theory and matching: to allow for individuals with incomplete preferences and multidivisional organizations.

# Individual Choices

- provide weaker conditions giving rise to wMCS of individual choices than MS
- provide characterizations along the lines of [Milgrom and Shannon \(1994\)](#) and [Quah and Strulovici \(2007\)](#)

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— Omitted due to time constraint

# Pareto Optimal Choices

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- $I$ : finite set of individuals
- $X$ : set of possible (social) choices; a *poset* with  $\geq$
- $u_i : X \rightarrow \mathbb{R}$  payoff function for  $i \in I$ ;  
     $u = (u_i)$  profile of payoff functions
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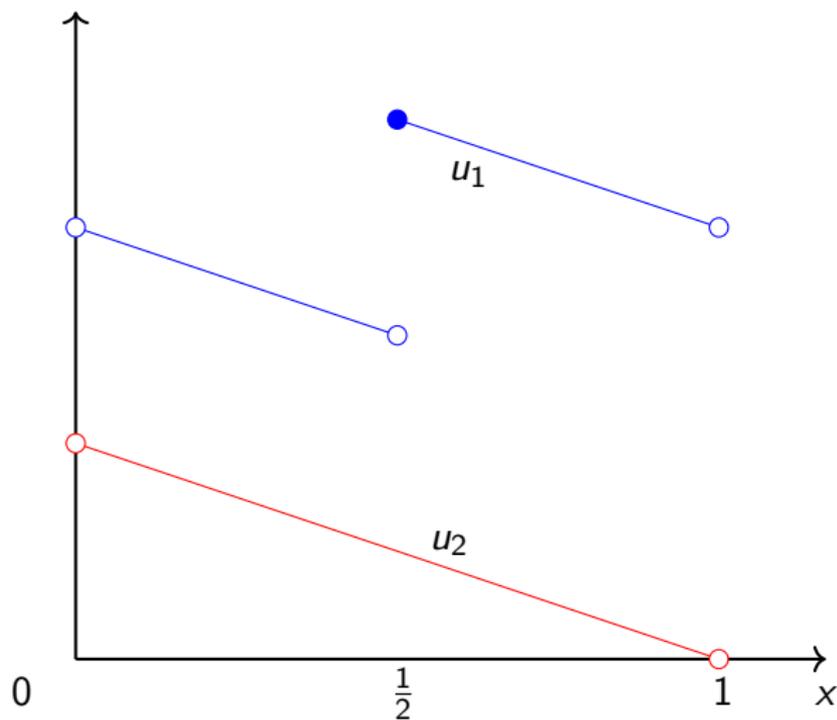
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- *Does MS condition for individuals imply wMCS of POCs?* **Not without additional condition.**

## Example

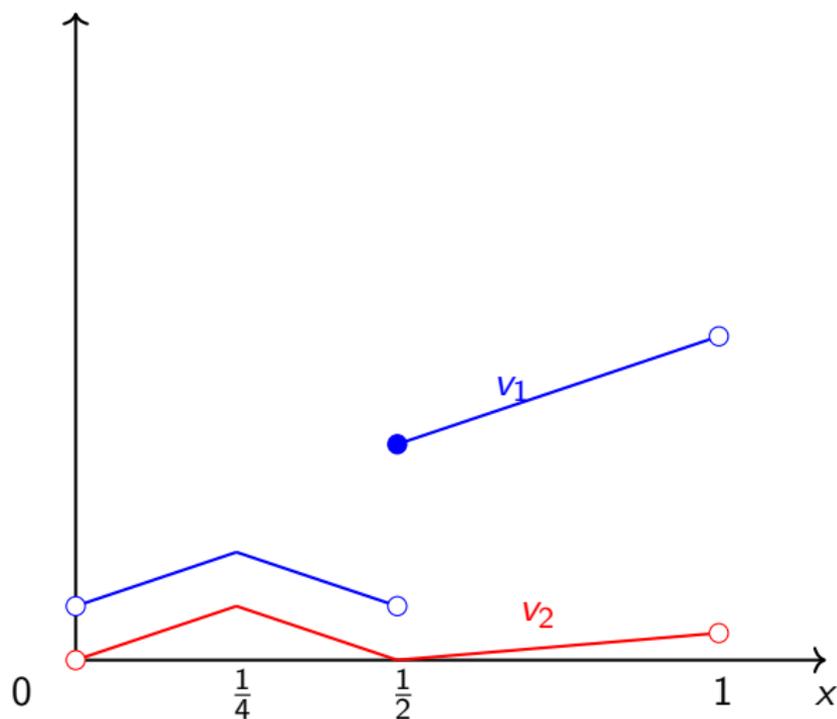
- Suppose  $X = (0, 1)$ .



- Unique Pareto optimum  $= \frac{1}{2}$ .

## Example: after a single crossing dominating shift

- Suppose  $X = (0, 1)$ .



- Unique Pareto optimum  $= \frac{1}{4}$  — Pareto optimum falls!!

## wMCS of POC: one-dimensional $X$

If  $X$  is totally ordered, the condition is simple:

### Theorem

Suppose

- (i)  $X$  is compact and  $u$  and  $v$  are upper semicontinuous;
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- In the example: If  $X = [0, 1]$ , then

$$P(u) = \{0, \frac{1}{2}\} \leq_{ws} \{\frac{1}{4}, 1\} = P(v).$$

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Similar argument shows  $\sup P(u) \leq \sup P(v)$ . With a little more care, the result follows.  $\square$

## wMCS of POC: General $X$

### Theorem

Suppose

- (i)  $X$  is a convex, compact lattice
- (ii)  $u$  and  $v$  are upper semicontinuous, concave, supermodular; and  $v$  increasing-difference dominates  $u$ .

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- **Supermodularity**: cardinal strengthening of quasi-supermodularity
- **Increasing differences**: cardinal strengthening of single crossing
- Upshot: Conditions guaranteeing sMCS for individual choices give wMCS for POCs, in a “well-behaved” environment.

## Proof Sketch

We utilize our new characterization of POC via a **sequential utilitarian welfare maximization**.

Theorem (Che, Kim, Kojima and Ryan, 2020)

Given our conditions,  $x \in P(u)$  if and only if there exists a sequence  $\{\phi^k\}_{k=1}^K$  of nonnegative welfare weights,  $\phi^k$  strictly positive, such that  $x \in X^k(u)$  for all  $k = 1, \dots, K$ , where

$$X^0(u) := X \text{ and } X^k(u) := \arg \max_{x' \in X^{k-1}(u)} \sum_i \phi_i^k u_i(x'). \quad \Rightarrow$$

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- The result then follows since

$$P(u) = \bigcup_{\{\phi^k\}} P_{\{\phi^k\}}(u) \leq_{ws} \bigcup_{\{\phi^k\}} P_{\{\phi^k\}}(v) = P(v).$$

(Strong set order is NOT closed under  $\cup$ , but weak set order is.)  $\square$

## Example

Let  $X = [0, 6]^2$ ,  $I = \{1, 2\}$  and

$$u_1(x, y) = -(x - 1)^2 - (y - 1)^2, \quad u_2(x, y) = -(x - 4)^2 - (y - 1)^2$$
$$v_1(x, y) = -(x - 1)^2 - (y - 4)^2, \quad v_2(x, y) = -(x - 4)^2 - (y - 2)^2.$$

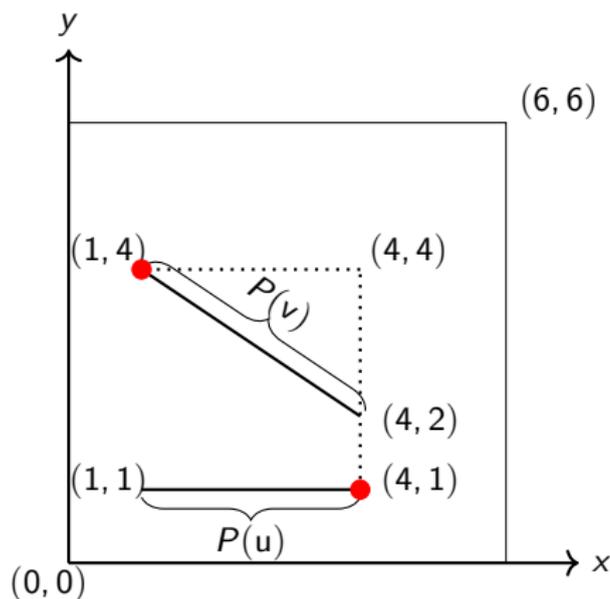


Figure: Failure of strong set monotonicity

# Fixed Point Theorem and Applications

# Tarski-Zhou Fixed Point Theorem

## Theorem (Tarski-Zhou)

Suppose

- $X$ : a complete lattice
- $F : X \rightrightarrows X$ : non-empty, complete sublattice-valued, strong set monotonic

Then, the fixed point set is nonempty and a complete lattice.

# New Fixed Point Theorem

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## Theorem (Li-CKK)

Suppose

- $X$ : **partially ordered**, and **compact**
- $F : X \rightrightarrows X$ : non-empty, **compact-valued**, (upper) weak set monotonic
- **regularity**:  $X_+(F) := \{x \in X : \exists y \geq x \text{ s.t. } y \in F(x)\}$  is non-empty.

Then, the fixed point set is nonempty and contains a maximal point.

- **Note**: analogous for “lower weak set monotonicity”

# Comparison

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## wMCS of Fixed Point Set

Let  $\mathcal{E}(F)$  be the fixed point set of  $F$ .

### Theorem (CKK)

Suppose  $X$  is compact, both  $F$  and  $G$  satisfy CKK conditions. If  $G(x) \geq_{uws} F(x)$  for all  $x$ , then  $\mathcal{E}(G) \geq_{uws} \mathcal{E}(F)$ .

- analogous for “lower weak set monotonic.”

### Theorem

With order continuity (satisfied if  $X$  is finite), a fixed point can be found iterating  $F$  from a regular point (i.e.,  $X_+$  or  $X_-$ ).

- But, can't guarantee obtaining a maximal or minimal fixed point this way.  $\Rightarrow$

# Application: Games with Weak Strategic Complementarities

- $\Gamma = (I, X, (B_i)_{i \in I})$  a game where
  - ▶  $I$ : finite set of players
  - ▶  $X$ : set of strategy profiles
  - ▶  $B_i$ : best response correspondence for player  $i$
- $\Gamma$  is a **game with weak strategic complementarities** if
  - ▶ for each  $i$ ,  $B_i$  is nonempty, compact-valued and upper (or lower) weak set monotonic;
  - ▶  $B = (B_i)_{i \in I}$  satisfies regularity.

## wMCS of Nash equilibria

### Theorem

- 1 A game  $\Gamma$  with weak strategic complementarities has a nonempty set of Nash equilibria.
- 2 Suppose that  $\Gamma'$  and  $\Gamma$  are both games with weak strategic complementarities, and that  $B'_i(s_{-i}) \geq_{uws} B_i(s_{-i})$  for every  $i \in I$  and  $s_{-i} \in S_{-i}$ . Then,  $\mathcal{NE}(\Gamma') \geq_{uws} \mathcal{NE}(\Gamma)$ .

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- Requirement weaker than standard “(quasi)supermodular” games (Milgrom and Shannon (1994))
- Preferences need not be complete:  $B_i$  can simply be Pareto optimal choices (recall results before)
  - ▶ Can apply this result to **games played by coalitions** in which each coalition selects POCs for its members.

## Example: Generalized Bertrand game

### Generalized Bertrand game:

A set  $I$  of firms choose prices  $p_i \in P_i$ , for each  $i \in I$ , where  $P_i \subset \mathbb{R}_+$  is **finite**. Each firm faces a constant marginal cost  $c_i \geq 0$ , and demand function  $D_i : P_i \times P_{-i} \rightarrow \mathbb{R}_+$  that satisfies:

- (i)  $D_i$  is nonincreasing in  $p_i$  and nondecreasing in  $p_{-i}$ ;
- (ii) For any  $p_i < p'_i$ ,  $p_{-i} < p'_{-i}$ , and  $D(p_i, p_{-i}) > 0$ ,

$$\frac{D_i(p'_i, p_{-i})}{D_i(p_i, p_{-i})} \leq \frac{D_i(p'_i, p'_{-i})}{D_i(p_i, p'_{-i})}.$$

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$$\frac{D_i(p'_i, p_{-i})}{D_i(p_i, p_{-i})} \leq \frac{D_i(p'_i, p'_{-i})}{D_i(p_i, p'_{-i})}.$$

- Condition (ii) weaker than the existing condition (e.g., [Milgrom and Shannon \(1994\)](#)), whereby  $D_i > 0$  and is smooth, and satisfies log supermodularity.
- Includes pure Bertrand game (which existing conditions does not include).

## Example: continued

Best response correspondence is lower weak set monotonic.  $\Rightarrow$  a game of weak strategic complementarities.

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### Proposition

For any vector  $c$  of marginal costs, the set of NE  $\mathcal{NE}(c)$  is nonempty. For  $c < c'$ ,  $\mathcal{NE}(c) \leq_{lws} \mathcal{NE}(c')$ .

## Application: General Model of Two Sided Matching with Contracts

- $W$ : finite set of workers
- $F$ : finite set of firms
- $X$ : finite set of contracts; a contract  $x \in X$  specifies a worker  $w$  and a firm  $f$  and a contract term (salary).
- **choice correspondence**:  $C_a(X')$  are optimal choices by agent  $a \in F \cup W$  from  $X'$ : *a nonempty family of subsets of  $X'$ .*
- **stable allocation** suitably defined—*Individually Rational and No Blocking.*

## Conditions on $C_a$

- ① **Weak Substitutability:** the rejection correspondence  $R_a(X') = \{Z : Z = X'_a \setminus Y \text{ for some } Y \in C_a(X')\}$  is **weak set monotonic** with “ $\supset$ ” as order.

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- ② **Sen's  $\alpha$ :**  $Y \in C_a(X'')$  and  $Y \subset X' \subset X'' \Rightarrow Y \in C_a(X')$ .
  - ▶ Weaker than **WARP** = **Sen's  $\alpha$**  + **Sen's  $\beta$** .
  - ▶ **Sen's  $\beta$ :**  $Y, Y' \in C_a(X'), Y \in C_a(X''), X' \subset X'' \Rightarrow Y' \in C_a(X'')$
  - ▶ Relaxing Sen's  $\beta$  accommodates **incomplete preferences**  $\Rightarrow$
- cf. State of the art assumes a stronger version of 1 and WARP.

## Fixed Point Characterization of Stability

Build a tâtonnement-like operator:  $T(X', X'') = (T_1(X''), T_2(X'))$ , for each  $(X', X'') \in 2^X \times 2^X$ , where

$$T_1(X'') = \{\tilde{X} \in 2^X : \tilde{X} = X \setminus \tilde{Y} \text{ for some } \tilde{Y} \in R_W(X'')\},$$

$$T_2(X') = \{\tilde{X} \in 2^X : \tilde{X} = X \setminus \tilde{Y} \text{ for some } \tilde{Y} \in R_F(X')\},$$

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where  $R_W$  and  $R_F$  defined similarly to before.

### Theorem

Suppose  $C_a$  satisfies Sen's  $\alpha$  for all  $a$ . Then,  $Z$  is stable if and only if there exists a fixed point  $(X', X'')$  of  $T$  such that  $Z \in C_F(X') \cap C_W(X'')$ .

- cf. The state of art assumes WARP.

# Existence of Stability

## Theorem

Suppose choice correspondences satisfy Sen's  $\alpha$  and weak substitutability. Then, a stable allocation exists.

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Suppose choice correspondences satisfy Sen's  $\alpha$  and weak substitutability. Then, a stable allocation exists.

## Proof Sketch.

- Define a partial order set  $(2^X \times 2^X, \geq)$  with  $\geq = (\supset, \subset)$ .
- Weak Substitutability:  $T$  is weak set monotonic.
- Fixed Point Theorem:  $T$  has a fixed point

By our characterization, a stable allocation exists. □

**Remark:** Gale-Shapley is an iterative version of Tarski that works for a simple environment. We are generalizing it.

## weak MCS

### Theorem

Suppose that a firm's choice correspondence becomes more permissive (in set inclusion). Then, workers become better off and firms become worse off in the weak set order sense (under original preferences).

## weak MCS

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Suppose that a firm's choice correspondence becomes more permissive (in set inclusion). Then, workers become better off and firms become worse off in the weak set order sense (under original preferences).

### Proof Sketch.

- Stable allocation = Fixed point of  $T$
- Change in choice  $\Rightarrow$  Change in  $T$
- Use Comparative statics of fixed points



# Applications:

- ① Multidivisional organizations
- ② Matching with Regional Constraints

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- Corollaries: Existence of stable allocations, comparative statics: when the hiring constraint becomes more restrictive; all other firms benefit, workers are hurt.

# Conclusion

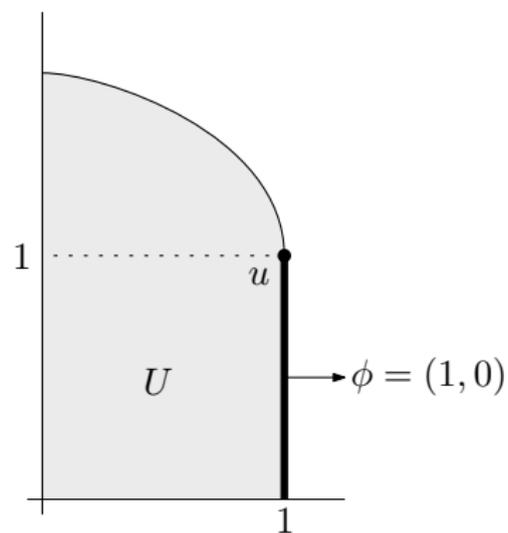
- We propose **weak monotone comparative statics (wMCS)**
- Requirement is weaker, so wider applicability
- Analyzed: individual choices, Pareto optimal choices, games with weak strategic complementarity, matching theory
- Future Research:
  - ▶ Weaker sufficient conditions for wMCS of Pareto optimal choices
  - ▶ More applications

Thank You!

# Some References

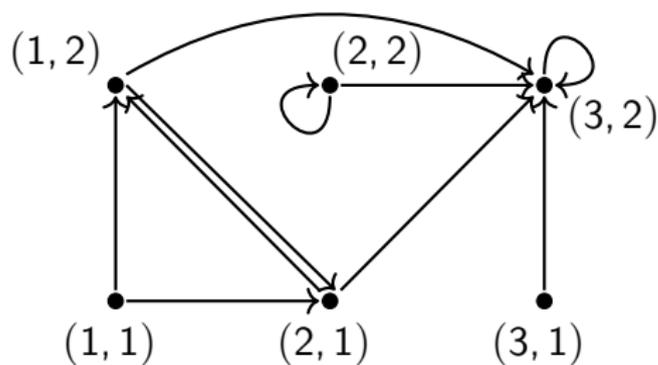
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# Illustration of Non-Exposed Pareto Optimum



- $u$  is Pareto optimal but not exposed.

## Failure of any iteration to reach a minimal fixed point



- The minimal fixed point  $(2,2)$  cannot be reached from any iterative application of  $F$  starting from  $(1,1)$ .

◀ Return

## Violation of Sen's $\beta$ due to Preference Incompleteness

- A firm  $f$  with two divisions,  $\delta$  and  $\delta'$ , and three workers  $w$ ,  $w'$ , and  $w''$ .
- Workers are all acceptable to  $\delta$  and  $\delta'$  while  $w'' \succ_{\delta'} w'$ .
- Constrained to hire at most one worker across the divisions.
- No strict preferences over which division should hire a worker when both divisions have applicants.
- $C_f(\{(w, \delta), (w', \delta')\}) = \{\{(w, \delta)\}, \{(w', \delta')\}\}$ .
- But  $C_f(\{(w, \delta), (w', \delta'), (w'', \delta')\}) = \{\{(w, \delta)\}, \{(w'', \delta')\}\}$ .

◀ Return

Li (2014), Fleiner (2003), Che et al. (2020), Tarski (1955), Zhou (1994)